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On cardinal invariants and metrizability of topological inverse semigroups

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Abstract

Let S be a topological inverse semigroup, $E=\{x\in S\colon\ xx=x\}$ be the maximal semilattice in S, and $C=\{x\in S\colon\ xe=ex$ for every idempotent $e\in E\}$ be the maximal Clifford semigroup of S. It is proven that a Lindelöf locally compact semigroup S is metrizable if and only if the maximal Clifford semigroup C is metrizable. We derive from this that a compact topological inverse semigroup S is metrizable, provided the maximal semilattice E and all maximal groups of S are metrizable and one of the following conditions is satisfied:

(1) (MA+¬CH) holds;
(2) E is a G_δ-set in the maximal Clifford semigroup C of S;
(3) E is a Lawson semilattice;
(4) all maximal groups of C are Lie groups;
(5) S is dyadic or scadic compact;
(6) S is a fragmentable (or Rosenthal) monolithic compactum;
(7) S is a Corson (or Rosenthal) compactum with countable spread.

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In this note we proceed investigations of cardinal invariants and metrizability of topological inverse semigroups started [4,8,9,3] and answer some questions posed in those papers.

First we remind necessary definitions. A set S equipped with an associative operation $*: S \times S \to S$ is called an *inverse semigroup* if for every element $x \in S$ there exists a unique element of S—denoted by x^{-1} and called *the inverse element* of x—such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. An inverse semigroup S is called an *inverse Clifford semigroup* if $xx^{-1} = x^{-1}x$ for every $x \in S$. A *semilattice* is a set S endowed with an associative commutative operation $*: S \times S \to S$ such that each element x of S is an *idempotent*, that is $x * x = x$. Clearly, each semilattice is an inverse Clifford semigroup; also, an inverse semigroup is a group if and only if it has a unique idempotent.

If an inverse semigroup S is given with a topology such that the maps $*: S \times S \to S$ and $(\cdot)^{-1}: S \to S$ are continuous, then S is called a *topological inverse semigroup*. All topological spaces considered in this paper are Hausdorff.

Let S be a topological inverse semigroup. Denote by E the set of all idempotents of S. Clearly, the set E is closed in S. Moreover, there are two natural retractions $\pi_1: S \to E$ and $\pi_2: S \to E$ defined by $\pi_1(x) = xx^{-1}$ and $\pi_2(x) = x^{-1}x$ for $x \in S$. It is known that the restriction of the semigroup operation on E is commutative [13], so E is a topological semilattice. In fact, E is the maximal semilattice in S. For an idempotent $e \in E$ let H_e be the maximal group in S containing the idempotent e. It can be shown that $H_e = \pi_1^{-1}(e) \cap \pi_2^{-1}(e)$.

Clearly, an inverse semigroup S is Clifford if and only if $\pi_1 = \pi_2$. In this case $S = \bigcup_{e \in E} H_e$, i.e., the inverse Clifford semigroup S decomposes onto groups parameterized by the set of idempotents.

In this context the following problem arises naturally (see [4,8,9]): *Suppose the maximal semilattice E of a topological inverse semigroup S has a topological property $\mathcal{P}_1$, while all maximal groups H_e, $e \in E$, of S have a topological property $\mathcal{P}_2$. What can be said about topological properties of the space S*?

As a particular case of this problem we will consider the following

Metrization Conjecture. *A compact topological inverse semigroup S is metrizable if and only if the maximal semilattice E and all maximal groups H_e, $e \in E$, of S are metrizable.*

For compact topological inverse *Clifford* semigroups this conjecture is independent of ZFC-axioms (see [3]): it is true under (MA+¬CH) and false under CH. In this paper we show that the above metrization conjecture is equivalent to the corresponding metrization conjecture for compact topological inverse Clifford semigroups and thus is independent of ZFC axioms too.

The reduction is based on a simple observation. Given an inverse semigroup S let $C = \{x \in S\colon\ xe = ex$ for every idempotent $e \in E\} \subset S$. It is known that an inverse semigroup X is Clifford if and only if $xe = ex$ for every $x \in X$ and every idempotent e in X, see [13]. This yields that C is a maximal Clifford subsemigroup in S. In the sequel, we call C *the maximal Clifford semigroup* of S. We remark that C, being a subset of $\bigcup_{e \in E} H_e$, needs not coincide with $\bigcup_{e \in E} H_e$ (see Example 8 below).

It turns out that a compact topological inverse semigroup S is metrizable if and only if the maximal Clifford semigroup C of S is metrizable. This fact together with results of [3] yields the following metrization theorem: a compact topological inverse semigroup S is metrizable, provided the maximal semilattice E and all maximal groups of the Clifford semigroup C are metrizable and one of the following conditions is satisfied:

(1) (MA+¬CH) holds;
(2) E is a G_δ-set in C;
(3) E is a Lawson semilattice;
(4) all maximal groups of C are Lie groups;
(5) C or S is dyadic or scadic compact;
(6) C or S is a fragmentable (or Rosenthal) monolithic compactum;
(7) C or S is a Corson (or Rosenthal) compactum with countable spread.

We start with investigating cardinal invariants of topological inverse semigroups. Let us recall that for a topological space X

- the *weight* $w(X)$ is the minimal cardinality of a base of the topology of the space X;
- the *density* $d(X)$ is the minimal cardinality of a dense subset of X;
- the *Lindelöf number* $l(X)$ is the smallest cardinal τ such that any open cover $\mathcal{U}$ of X has a subcover $\mathcal{V}$ with $|\mathcal{V}| \leqslant \tau$;
- the *spread* or *hereditary cellularity* $hc(X) = \sup\{|D|\colon\ D$ is a discrete subspace of $X\}$;
- the *tightness* $t(X)$ is the smallest cardinal τ such that for every subset $A \subset X$ and every point a of the closure $\overline{A}$ of A in X there exists a subset $B \subset A$ such that $|B| \leqslant \tau$ and $\overline{B} \ni a$;
- the *character* $\chi(x, X)$ *at a point* $x \in X$ is the minimal cardinality of a neighborhood base at x;
- the *character* $\chi(X) = \sup\{\chi(x, X)\colon\ x \in X\}$;
- the π*-character* $\pi\chi(x, X)$ *at a point* $x \in X$ is the smallest size of a collection $\mathcal{U}$ of nonempty open subsets of X such that each neighborhood of x contains an element of the collection $\mathcal{U}$;
- the π*-character* $\pi\chi(X) = \sup\{\pi\chi(x, X)\colon\ x \in X\}$;
- the *pseudocharacter* $\psi(A, X)$ of a subset $A \subset X$ is the smallest size of a collection $\mathcal{U}$ of open subsets of X such that $A = \bigcap \mathcal{U}$;
- the *pseudocharacter* $\psi(X) = \sup\{\psi(\{x\}, X)\colon\ x \in X\}$;
- the *diagonal number* $\Delta(X) = \psi(\Delta_X, X \times X)$, where $\Delta_X = \{(x, x)\colon\ x \in X\}$ is the diagonal of $X \times X$;
- the *Shapirovskiĭ number* $\rho(X) = \aleph_0 \cdot \sup\{\tau\colon$ there is a continuous surjective map $f : X \to [0, 1]^\tau\}$.

Besides purely topological cardinal invariants there are certain cardinal functions depending on the algebraic structure of a topological inverse semigroup. For a topological inverse semigroup S let $ib(S)$, the *index of boundedness* of S be the smallest infinite cardinal τ such that the semigroup S is τ*-bounded*. The latter means that for any neighborhood U of the maximal semilattice E of S there is a subset $F \subset S$ such that

$|F| \leqslant \tau$ and $S = F \cdot U$. The index of boundedness turned to be very useful in the theory of topological groups, see [17]. For a semilattice E by $\max E$ the set of all maximal elements of E is denoted.

It is well known that for any topological group G its cardinal invariants relate as follows: $ib(G) \leqslant \min\{c(G), l(G)\}$, $w(G) = \pi w(G) = ib(G) \cdot \chi(G)$, $nw(G) \leqslant k(G) \cdot \psi(G)$, $c(G) \leqslant k(G) \cdot \aleph_0$, and $\chi(G) = \pi\chi(G)$, see [1,2] or [17, §4]. Moreover, for a compact topological group G we have the equalities $\aleph_0 = c(G) = sh(G) \leqslant d(G) \leqslant \chi(G) = \pi\chi(G) = \psi(G) = \Delta(G) = t(G) = hl(G) = hc(G) = \rho(G) = nw(G) = \pi w(G) = w(G)$ the most of which hold for any dyadic compactum, see [6, 3.12.12].

The situation with cardinal invariants of topological inverse semigroups is much more complex (even in the compact case). We summarize all known (positive) information in the following theorem.

Theorem 1. *Let S be a topological inverse semigroup, E be the maximal semilattice of S, C be the maximal Clifford semigroup of S, and H_e, $e \in E$, are maximal groups of S. Then*

(1) $\psi(S) = \sup\{\psi(E), \psi(H_e)\colon e \in E\}$;
(2) $\Delta(S) \leqslant \max\{\Delta(E), \psi(E, S)\}$;
(3) $\psi(E, S) \leqslant \max\{\psi(E, C), \psi(C, S)\}$;
(4) $\psi(C, S) \leqslant \max\{\Delta(E), d(E)\}$;
(5) $|\max E| \leqslant ib(S) \leqslant \sup\{|E|, ib(H_e)\colon e \in E\}$;
(6) $t(S) = \sup\{t(E), t(H_e)\colon e \in E\}$, *provided the retractions π_1, π_2 are closed maps*;
(7) $w(S) = \max\{l(S), \Delta(E), d(E), \psi(E, C)\} = l(S) \cdot w(C) = hl(S) \cdot \Delta(E)$ *and* $\pi\chi(S) \leqslant \chi(S) = \sup\{\chi(E), \pi\chi(H_e)\colon e \in E\}$, *provided S is a locally compact space*;
(8) $\chi(S) = \psi(S) = t(S) \cdot \chi(E) = \rho(S) \cdot \chi(E) \leqslant hc(S) \cdot \chi(E)$ *if S is compact.*

Proof. (1) The inequality $\psi(S) \geqslant \sup\{\psi(E), \psi(H_e)\colon e \in E\}$ is trivial. To prove the inverse inequality, fix a point $x \in S$. Let $e = \pi_1(x) = xx^{-1}$, $f = \pi_2(x) = x^{-1}x$ and let $H_{e,f} = \pi_1^{-1}(e) \cap \pi_2^{-1}(f) \subset S$. Let $\mathcal{U}_1$, $\mathcal{U}_2$ be collections of open subsets of E such that $|\mathcal{U}_1|, |\mathcal{U}_2| \leqslant \psi(E)$, $\bigcap \mathcal{U}_1 = \{e\}$, and $\bigcap \mathcal{U}_2 = \{f\}$. Then $\mathcal{W}_1 = \{\pi_1^{-1}(U)\colon U \in \mathcal{U}_1\}$, $\mathcal{W}_2 = \{\pi_2^{-1}(U)\colon U \in \mathcal{U}_2\}$ are collections of open subsets of S such that $|\mathcal{W}_1|, |\mathcal{W}_2| \leqslant \psi(E)$ and $(\bigcap \mathcal{W}_1) \cap (\bigcap \mathcal{W}_2) = H_{e,f}$.

It is well known that the spaces $H_{e,f}$ and H_e are homeomorphic. Indeed, the map $h\colon H_e \to H_{e,f}$ defined by $h(y) = yx$ for $y \in H_e$ is a homeomorphism with the inverse h^{-1} acting as $h^{-1}(z) = zx^{-1}$ for $z \in H_{e,f}$. Hence $\psi(H_{e,f}) = \psi(H_e)$ and there exists a collection $\mathcal{W}_0$ of open subsets in S such that $|\mathcal{W}_0| \leqslant \psi(H_{e,f}) = \psi(H_e)$ and $H_{e,f} \cap (\bigcap \mathcal{W}_0) = \{x\}$. Finally, letting $\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1 \cup \mathcal{W}_2$, we see that $\mathcal{W}$ is a collection of open subsets in S such that $|\mathcal{W}| \leqslant \max\{\psi(E), \psi(H_e)\}$ and $\bigcap \mathcal{W} = \{x\}$. Thus $\psi(S) \leqslant \sup\{\psi(E), \psi(H_e)\colon e \in E\}$.

(2) To verify that $\Delta(S) \leqslant \max\{\Delta(E), \psi(E, S)\}$, fix a collection $\mathcal{U}$ of open subsets in S such that $|\mathcal{U}| \leqslant \psi(E, S)$ and $\bigcap \mathcal{U} = E$, and a collection $\mathcal{V}$ of open sets in $E \times E$ such that $|\mathcal{V}| \leqslant \Delta(E)$ and $\bigcap \mathcal{V} = \Delta_E = \{(e, e)\colon e \in E\} \subset E \times E$.

For every $U \in \mathcal{U}$ and $V \in \mathcal{V}$ let

$$W_U = \left\{(x, y) \in S \times S\colon xy^{-1} \in U\right\} \quad \text{and}$$

$$W_V = \big\{(x, y) \in S \times S\colon \big(x^{-1}x, y^{-1}y\big) \in V\big\}.$$

Clearly, the sets W_U and W_V are open in $S \times S$ and the collection $\mathcal{W} = \{W_U\colon U \in \mathcal{U}\} \cup \{W_V\colon V \in \mathcal{V}\}$ has size $|\mathcal{W}| \leqslant \max\{\Delta(E), \psi(E, S)\}$. We claim that $\bigcap \mathcal{W} = \Delta_S$. Indeed, let $(x, y) \in \bigcap \mathcal{W}$. Then $xy^{-1} = e \in E$ and $x^{-1}x = y^{-1}y$. Observe that $x = xx^{-1}x = xy^{-1}y = ey$. On the other hand, $e = e^{-1} = (xy^{-1})^{-1} = yx^{-1}$ implies $y = yy^{-1}y = yx^{-1}x = ex = e(ey) = ey = x$. Thus $(x, y) \in \Delta_S$ and $\Delta(S) \leqslant |\mathcal{W}| \leqslant \max\{\Delta(E), \psi(E, S)\}$.

(3) The inequality $\psi(E, S) \leqslant \max\{\psi(E, C), \psi(C, S)\}$ is trivial.

(4) To show that $\psi(C, S) \leqslant \max\{\Delta(E), d(E)\}$, fix a dense subset $D \subset E$ with $|D| = d(E)$ and a collection $\mathcal{U}$ of open subsets in $E \times E$ with $|\mathcal{U}| = \Delta(E)$ and $\bigcap \mathcal{U} = \Delta_E = \{(e, e)\colon e \in E\} \subset E \times E$. Using the uniqueness of the inverse element in the semigroup S, one may verify that

$$C = \big\{x \in S\colon \forall e \in E,\ \pi_1(xe) = \pi_2(ex) = e\pi_1(x) = e\pi_2(x)\big\}$$

and by the continuity of the retractions π_1, π_2,

$$C = \big\{x \in S\colon \forall e \in D,\ \pi_1(xe) = \pi_2(ex) = e\pi_1(x) = e\pi_2(x)\big\}.$$

For every $U \in \mathcal{U}$ and $e \in D$ let

$$W(U, e) = \big\{x \in S\colon \big\{\big(\pi_1(xe), \pi_2(ex)\big), \big(e\pi_1(x), e\pi_2(x)\big), \big(\pi_2(ex), e\pi_1(x)\big)\big\} \subset U\big\}.$$

Evidently, $\mathcal{W} = \{W(U, e)\colon U \in \mathcal{U},\ e \in D\}$ is a collection of open subsets in S such that $|\mathcal{W}| \leqslant \max\{\Delta(E), d(E)\}$ and $\bigcap \mathcal{U} = C$.

(5) To show that $ib(S) \leqslant \tau$ where $\tau = \sup\{|E|, ib(H_e)\colon e \in E\}$, fix any neighborhood $U \subset S$ of E. For every $e \in E$ find a subset $F_e \in H_e$ such that $|F_e| \leqslant \tau$ and $H_e \subset F_e \cdot U$. Next, for idempotents $e, f \in E$ fix a point $h_{e,f} \in H_{e,f}$ if $H_{e,f} \neq \emptyset$ and let $h_{e,f}$ be any point of S if $H_{e,f} = \emptyset$. It is clear that the set $F = \bigcup_{e,f \in E} h_{e,f} F_e$ has size $|F| \leqslant \tau$. We claim that $S = F \cdot U$. Indeed, given a point $x \in S$, let $e = xx^{-1}$ and $f = x^{-1}x$. Then $x \in H_{e,f} = h_{e,f} H_e \subset h_{e,f} F_e U \subset F \cdot U$.

To verify the inequality $|\max E| \leqslant ib(S)$, fix any subset $F \subset S$ with $|F| \leqslant \tau$ and $S = FS$. The inequality $|\max E| \leqslant ib(S)$ will follow as soon as we prove that $\max E \subset \pi_1(F)$. Fix any maximal element $e \in \max E$ and find points $x \in F$ and $y \in S$ with $e = xy$. Consider the idempotent $f = xx^{-1} \in \pi_1(F)$ and note that $fe = xx^{-1}xy = xy = e$. Then $e \leqslant f$ and by the maximality of e, we get $e = f \in \pi_1(F)$.

(6) Suppose the retractions $\pi_1, \pi_2 : S \to E$ are closed maps. First we prove that $t(\pi_1^{-1}(e)) \leqslant \max\{t(E), t(H_e)\}$ for every $e \in E$. As we have already remarked, for every $f \in E$ the space $H_{e,f} = \pi_1^{-1}(e) \cap \pi_2^{-1}(f)$ either is empty or is homeomorphic to H_e. Thus $t(H_{e,f}) \leqslant t(H_e)$ for every $f \in E$. Since the map $\pi_2 : \pi_1^{-1}(e) \to E$ is closed, we may apply [6, 3.12.8(d)] to conclude that $t(\pi_1^{-1}(e)) \leqslant \sup\{t(E), t(\pi_1^{-1}(e) \cap \pi_2^{-1}(f))\colon f \in E\} \leqslant \max\{t(E), t(H_e)\}$. Applying [6, 3.12.8(d)] once more (to the closed map π_1), we get $t(S) \leqslant \sup\{t(E), t(\pi_1^{-1}(e))\colon e \in E\} \leqslant \sup\{t(E), t(H_e)\colon e \in E\}$.

(7) The 7th statement follows immediately from the relations (1)–(4) and the well-known equalities $\chi(X) = \psi(X)$, $w(X) = \max\{l(X), \Delta(X)\}$ holding for each locally compact space X (see [6, 3.3.4] and [1, II.§1]) and the equality $\chi(H) = \pi\chi(H)$ holding for any topological group H, see [17, 4.3].

(8) Suppose that S is compact. We have to verify the equalities $\psi(S) = \chi(S) = t(S) \cdot \chi(E) = \rho(S) \cdot \chi(E) \leqslant hc(S) \cdot \chi(E)$. The first one holds because S is compact. It is known that $t(G) = \chi(G) = \psi(G) = hc(G) = \rho(G)$ for any compact topological group, see [2,16]. According to the items (6) and (7), we get $\chi(S) \geqslant \chi(E) \cdot t(S) = \sup\{\chi(E), t(H_e)\colon e \in e \in E\} = \sup\{\chi(E), \chi(H_e)\colon e \in E\} = \chi(S)$ and thus the second equality is also true.

According to [15], $\rho(Y) \leqslant \rho(X) \leqslant \chi(X)$ for any closed subspace Y of a compact topological space X. Then $\chi(S) \geqslant \chi(E) \cdot \rho(S) \geqslant \sup\{\chi(E), \rho(H_e)\colon e \in E\} = \sup\{\chi(E), \chi(H_e)\colon e \in E\} = \chi(S)$ and thus $\chi(S) = \rho(S) \cdot \psi(E)$.

Finally, observe that $\chi(S) = \sup\{\chi(E), \chi(H_e)\colon e \in E\} = \sup\{\chi(E), hc(H_e)\colon e \in E\} \leqslant \max\{\chi(E), hc(S)\}$. □

Now we apply the proven theorem to the metrization problem for topological inverse semigroups.

Following [7, 3.5] and [12], we define a regular topological space X to be an *M-space* if there is a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers of X such that (1) for each n, $\mathcal{U}_{n+1}$ star refines $\mathcal{U}_n$ and (2) if $x_n \in \mathcal{S}t(x, \mathcal{U}_n)$ for each $n \in \omega$, then the sequence (x_n) has a cluster point in X. As usual, $\mathcal{S}t(x, \mathcal{U}) = \bigcup\{U \in \mathcal{U}\colon x \in U\}$ for a point $x \in X$ and a cover $\mathcal{U}$ of X.

The class of M-spaces includes all metrizable spaces, all paracompact locally compact spaces and all countably compact spaces. Moreover, a paracompact space X is an M-space if and only if X is homeomorphic to a closed subspace of the product of a metric space and a compact space, see [7, §3]. According to [7, 3.8], an M-space X is metrizable if and only if $\Delta(X) \leqslant \aleph_0$. This criterion together with Theorem 1 implies

Corollary 2. *A regular topological inverse semigroup S is metrizable provided S is an M-space and the maximal semilattice E of S is a* (*separable*) *metrizable G_δ-set in S* (*in the maximal Clifford semigroup C of S*).

This corollary implies that a paracompact locally compact topological inverse semigroup S is metrizable if and only if the maximal semilattice is a metrizable G_δ-set in S. In fact, the same statement is true for the wider class of weakly paracompact locally compact topological inverse semigroups.

We recall that a space X is *paracompact* (respectively *weakly paracompact*) if for every open cover $\mathcal{U}$ of X there is a locally finite (respectively point-finite) open cover $\mathcal{V}$ of X, inscribed into the cover $\mathcal{U}$, see [6].

Corollary 3. *A weakly paracompact locally compact topological inverse semigroup S is metrizable provided the maximal semilattice E of S is a metrizable* (*separable*) *G_δ-set in S* (*in the maximal Clifford semigroup C of S*).

Proof. The "only if" part is trivial. To prove the "if" part, observe that E is a G_δ-set is S if E is a metrizable separable G_δ-set in C, see Theorem 1(4).

Then Theorem 1 yields $\Delta(S) \leqslant \max\{\Delta(E), \psi(E, S)\} \leqslant \aleph_0$. The space S, being a locally compact space with countable diagonal number $\Delta(S)$, is locally metrizable, see [1, II.§1]. Hence, S is a weakly paracompact locally separable space. By [6, 5.3.A], the space S is paracompact, and by [6, 5.4.A], S, being paracompact and locally metrizable, is metrizable.

Corollary 4. *Assume* (MA+¬CH). *A topological inverse semigroup S is metrizable provided S is an M-space, and the maximal semilattice E as well as all maximal groups of S are metrizable and separable.*

Proof. Suppose the semigroup S is an M-space such that the maximal semilattice E and all maximal groups H_e, $e \in E$, of S are metrizable and separable. The maximal Clifford semigroup C, being a closed subset in S, is an M-space. Denote by G_e, $e \in E$, the maximal groups of C. Clearly, $G_e = H_e \cap C \subset H_e$ for all $e \in E$. Consequently, all maximal groups G_e, $e \in E$, of C are metrizable and separable. Applying (MA+¬CH) and Theorem 3.5 of [3] we conclude that the Clifford semigroup C is metrizable. Consequently, E is a separable metrizable G_δ-set in C and by Corollary 2, the semigroup S is metrizable. □

Corollary 5. *Assume* (MA+¬CH). *A locally compact Lindelöf topological inverse semigroup S is metrizable if and only if the maximal semilattice E and all the maximal groups of S are metrizable.*

Recall that a *Lawson semilattice* is a topological semilattice admitting a base of the topology, consisting of subsemilattices, see [11]. A topological space X is *countably compact* if each open cover of X admits a finite subcover. The counterexamples constructed under CH in [3, §4] show that Corollaries 4 and 5 cannot be proven in ZFC. Nonetheless, in ZFC we have a positive partial result generalizing [3, 3.8 and 3.10] as well as Theorem 2 of [8] and Theorem 7 of [10].

Corollary 6. *A countably compact topological inverse semigroup S is metrizable provided one of the following conditions is satisfied*:

(1) *the maximal Clifford semigroup C of S is metrizable*;
(2) *the maximal semilattice E of S is a metrizable Lawson semilattice and all maximal groups of S are metrizable*;
(3) *the maximal semilattice E of S is metrizable and all maximal groups of S are Lie groups.*

Proof. (1) Suppose that the maximal Clifford semigroup C of S is metrizable. Then C, being a metrizable closed subspace of the countably compact space S, is compact and separable. Consequently, the maximal semilattice E of S is a metrizable separable G_δ-subset of C. Since each countably compact space is an M-space, we can apply Corollary 2 to conclude that the space S is metrizable.

(2) If E is a metrizable Lawson semilattice and all maximal groups of S are metrizable, then E, being a metrizable closed subspace of the countable compact space S, is compact and separable. The metrizability of the maximal groups of S implies the metrizability of the maximal groups of the Clifford semigroup C. Thus it is legal to apply Theorem 3.7 of [3] to conclude that the space C is metrizable. Now the preceding item finishes the proof.

(3) Suppose the maximal semilattice E is metrizable and all maximal groups H_e, $e \in E$, of S are Lie groups. Then $C \cap H_e$, $e \in E$, are maximal groups of the Clifford semigroup C. Since C is closed in S, each $C \cap H_e$, being a closed subgroup of the Lie group H_e, is a Lie group too. The semilattice, being a closed metrizable subspace of S, is compact and separable. Then by Theorem 3.10 of [3], the semigroup C is metrizable and by the first item, S is a metrizable space. □

A topological space X is called *scadic* if X is a continuous image of a product of compact scattered spaces (a space X is *scattered* if each subspace of X has an isolated point). Clearly, each dyadic compactum (that is a continuous image of a Cantor discontinuum $\{0,1\}^\tau$) is scadic. Like dyadic compacta, all first-countable scadic compacta are metrizable, see [5]. The same is true for π_χ*-spaces*, i.e., spaces X such that $w(\{x \in X\colon \pi\chi(x, X) < \tau\}) < \tau$ for every regular uncountable cardinal τ, see [14, §7].

Next, we remind the definitions of some classes of compacta appearing in functional analysis. A space X is *monolithic* if $nw(Y) = d(Y)$ for each subspace Y of X; X is *fragmentable* if there is a metric ρ on X such that every non-empty subspace contains a non-empty open subset of arbitrary small ρ-diameter; X is *Rosenthal compact* if X is homeomorphic to a compact subset of the space $B_1(P)$ of all functions of the first Baire class on a Polish space P; X is *Corson compact* if X is homeomorphic to a compact subset of a Σ-product of lines. The metrization problem for compact topological inverse Clifford semigroups that are scadic, dyadic, fragmentable, Corson or Rosenthal compact was considered in [3, §3]. Theorems 3.12 and 3.13 proved in [3] in combination with Theorem 1(8) and Corollary 6 imply the following Metrization Criterion.

Corollary 7. *The following conditions are equivalent for every compact topological inverse semigroup S with metrizable maximal semilattice E*:

(1) *the semigroup S is metrizable*;
(2) *the maximal Clifford semigroup C of S is metrizable*;
(3) *the space S is scadic or a π_χ-space and all maximal groups of S are first countable*;
(4) *S is a fragmentable* (*or Rosenthal*) *monolithic compactum*;
(5) *S is a Corson* (*or Rosenthal*) *compactum with countable spread.*

Next, we show that the maximal Clifford semigroup can be strictly smaller than the union of the maximal groups of an inverse semigroup.

Example 8. Consider the 7-element subsemigroup

$$I_7 = \{A, B, C, AA, BC, AB, AC\}$$

of the multiplicative semigroup of real (2×2)-matrices, where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is easy to verify that I_7 is an inverse semigroup, the maximal semilattice and the maximal Clifford semigroup of I_7 coincide with the set $\{B, C, BC, AA\}$ which is a proper subset of the union $\{A, B, C, BC, AA\}$ of all maximal groups of I_7.

In the subsequent proposition we will need the 5-element inverse subsemigroup $I_5 = \{BC, B, C, AB, AC\}$ of I_7. It can be shown that the maximal semilattice and the maximal Clifford semigroup of I_5 coincide with the set $I_3 = \{B, C, BC\}$, while all maximal groups of I_5 are trivial.

We remind that a topological space X is *supercompact* if it has an open subbase $\mathcal{S}$ such that every cover of X by members of $\mathcal{S}$ has a subcover consisting of no more than two elements; X is a *Moore space* if X is regular and admits a sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers such that the collection $\{\mathcal{S}t(x, \mathcal{U}_n)\}_{n\in\omega}$ forms a neighborhood base at each point $x \in X$.

The following proposition shows that the M-space or Lindelöf conditions in Corollaries 2–5 are essential.

Proposition 9. *There exists a non-metrizable topological inverse semigroup S such that*

(1) *S is a separable zero-dimensional locally compact locally metrizable Moore space admitting no supercompact compactification*;
(2) *S contains a closed discrete subspace of cardinality* $\mathfrak{c}$;
(3) *S is neither normal nor weakly paracompact nor an M-space*;
(4) *the maximal semilattice of S is a metrizable compact Lawson semilattice, open in S*;
(5) *the maximal Clifford semigroup of S coincides with the maximal semilattice of S*;
(6) *all maximal groups of S are trivial*;
(7) *S admits a continuous bijective semigroup homomorphism $S \to E \times I_5$ for some compact metrizable Lawson semilattice E.*

Proof. Let E be any metrizable compact uncountable zero-dimensional Lawson semilattice such that every non-maximal element of E is an isolated point in E (e.g., take E to be the usual binary tree). Clearly, the product $E \times I_5$ carries the structure of a compact metrizable topological inverse semigroup.

Now we define a new topology on $E \times I_5$. For an element $x \in E$ let $\downarrow x = \{y \in E \colon xy = y\}$ be the lower cone of x. Denote by $\widehat{E}$ the semilattice E endowed with the topology whose base consists of the sets $(\downarrow x) \cap U$, where $x \in E$ and U run over the topology of E. This topology was introduced and studied in details in [3, §1]. In particular, the space $\widehat{E}$ is separable, see [3, 1.5].

Let $I_3 = \{B, C, BC\}$ denote the maximal semilattice of the inverse semigroup I_5. Evidently, I_3 is also the maximal Clifford semigroup of I_5. Identifying $E \times I_5$ with the topological sum $S = E \times I_3 \cup \widehat{E} \times (I_5 \setminus I_3)$, we get the required topological inverse semigroup S. Clearly, the subset $E \times I_3$ coincides with the maximal semilattice and the maximal Clifford semigroup of S.

The properties (1)–(7) of the semigroup S can be established by analogy with the corresponding properties of the semigroup constructed in [3, 3.16]. $\square$

Finally we pose several problems suggested by the results of this paper.

Problem 10. What is the interplay between topological properties of a compact topological inverse semigroup S and those of the maximal Clifford semigroup $C \subset S$ and the maximal semilattice E? In particular,

(a) Is S countably cellular (or separable) if so is the space C?
(b) Is S countably cellular if the maximal semilattice E is second countable?
(c) Is S (hereditarily) separable if all maximal groups of S are (hereditarily) separable and the maximal semilattice is Lawson and (heriditarily) separable?
(d) Is S fragmentable (respectively Corson, Eberlein, Gul'ko, Radon-Nikodỳm, or Rosenthal) compact if so is the Clifford semigroup C?

Note that for compact topological inverse *Clifford* semigroups the answer to the questions (b), (c) are in positive, see [3].

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